A NEW FAMILY OF ELLIPTIC CURVES WITH POSITIVE RANK ARISING FROM PYTHAGOREAN TRIPLES

F.A.IZADI K.NABARDI F.KHOSHNAM

Abstract. The aim of this paper is to introduce a new family of elliptic curves in the form of $y^2 = x(x-a^2)(x-b^2)$ that have positive ranks. We first generate a list of pythagorean triples (a, b, c) and then construct this family of elliptic curves. It turn out that this new family have positive ranks and search for the upper bound for their ranks.

Keywords: elliptic curves; rank; pythagorean triples

AMS Classification: MSC2000.primary 14H52 ; Secondary 11G05, 14G05.

1. Introduction

An elliptic curve E over a field F is a curve that is given by an equation of the form

$$Y^2 + a_1XY + a_3 = X^3 + a_2X^2 + a_4X + a_6, \quad a_i \in F. \tag{1.1}$$

We let $E(F)$ denote the set of points $(x, y) \in F^2$ that satisfy this equation, along with a point at infinity denoted O [4].
In order for the curve (1.1) to be an elliptic it must be smooth, in other words, the three equations

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6, \tag{1.2}$$

$$a_1Y = 3X^2 + 2a_2X + a_4 \quad and \quad 2Y + a_1X + a_3 = 0$$

cannot be simultaneously satisfied by any $(x, y) \in E(\overline{F})$.

If $Char(F) \neq 2$, then we can reduce (1.1) to the following form

$$Y^2 = X^3 + aX^2 + bX + C \tag{1.3}$$

with the *discriminant* :

$$D = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2. \tag{1.4}$$

If furthermore the $Char(F)$ does not divide 6, then we get the simplest form of

$$Y^2 = X^3 + aX + b, \tag{1.5}$$

with the

$$D = -16(4a^3 + 27b^2). \tag{1.6}$$

Remark 1.1. The elliptic curve is smooth if and only if $D \neq 0$ [9].

2. Elliptic curves over Q

Mordell proved that on a rational elliptic curve, the rational points form a finitely generated abelian group, which is denoted by $E(Q)$ [4]. Hence we can apply the structure theorem for the finitely generated abelian groups to $E(Q)$ to obtain a decomposition of $E(Q) \cong Z^r \times Tors_E(Q)$, where r is an integer called the *rank* of E and $Tors_E(Q)$ is the finite abelian group consisting of all the elements of finite order in $E(Q)$.
In 1976, *Barry Mazur*, proved the following fundamental result:

$$\begin{array}{ll} \frac{Z}{mZ} & m = 1, 2, 3, ..., 10, 12 \\ \frac{Z}{2Z} \oplus \frac{Z}{mZ} & m = 2, 4, 6, 8 \end{array} \tag{2.1}$$

which shows that there is no points of order 11, and any $n \geq 13$.
There is an important theorem proved by *Nagell* and *Lutz*, which tells us how to find all of the rational points of finite order.

Theorem 2.1. *(Nagell-Lutz) Let E be given by $y^2 = x^3 + ax^2 + bx + c$ with $a, b, c \in Z$. Let $P = (x, y) \in E(Q)$. Suppose P has finite order, Then $x, y \in Z$ and either $y = 0$ or $y|D$.*

Proof. ([8] . *pp* . 56). □

Theorem 2.2. *Let E be given by $y^2 = x^3 + ax^2 + bx + c$ and, $P = (x, y) \in E(Q)$. P has an order 2 if and only if $y = 0$.*

Proof. ([9]. *pp* .77) . □

On the other hand, it is not known which values of *rank* r are possible. The current record is an example of elliptic curve over Q with *rank* ≥ 28 found by *Elkies* in may 2006 [2].

In this Paper we first introduce a family of elliptic curves over Q and show that they have positive rank, then search for the largest ranks possible.

3. Pythagorean triples

A primitive pythagorean triple is a triple of numbers (a, b, c) so that a , b and c have no common divisors and satisfy

$$a^2 + b^2 = c^2. \tag{3.1}$$

It's not hard to prove that if one of a or b is odd then the other is even, then c is always odd.
In general , we can generate (a, b, c) by the following relations:

$$a = i^2 - j^2 \qquad b = 2ij \qquad c = i^2 + j^2 \tag{3.2}$$

where $(i, j) = 1$ and i, j have oppositive parity.
The other way to generate (a, b, c) is the following forms:

$$a = \frac{i^2 - j^2}{2} \qquad b = ij \qquad c = \frac{i^2 + j^2}{2} \tag{3.3}$$

where $i > j \geq 1$ are chosen to be odd integers with no common factors [7].

The following table gives all possible triples with $i, j < 10$.

i	j	$a = i^2 - j^2$	$b = 2ij$	$c = i^2 + j^2$	(a, b, c)
2	1	3	4	5	$(3, 4, 5)$
3	2	5	12	13	$(5, 12, 13)$
4	1	15	8	17	$(15, 8, 17)$
4	3	7	24	25	$(7, 24, 25)$
5	2	21	20	29	$(21, 20, 29)$
5	4	9	40	41	$(9, 40, 41)$
6	1	35	12	37	$(35, 12, 37)$
6	5	11	60	61	$(11, 60, 61)$
7	2	45	28	53	$(45, 28, 53)$
7	4	33	56	65	$(33, 56, 65)$
7	6	13	84	85	$(13, 84, 85)$
8	1	63	16	65	$(63, 16, 65)$
8	3	55	48	73	$(55, 48, 73)$
8	5	39	80	89	$(39, 80, 89)$
8	7	15	80	113	$(15, 80, 113)$
9	2	77	36	85	$(77, 36, 85)$
9	4	65	72	97	$(65, 72, 97)$
9	8	17	144	145	$(17, 144, 145)$

TABLE 1. Generation pythagorean triples by i, j in range10

4. Structure Of The Curves

First we generate a list of pythagorean triples (a, b, c) with $i, j \leq 1000$. This yields a list of 202461 triples. Each (a, b, c) gives rise to the elliptic curve in the form

$$y^2 = x(x - a^2)(x - b^2). \tag{4.1}$$

Then we compute the $2-$ *selmer ranks* of these curves as upper bounds on the *Mordell* $-$ *Weil ranks*, finally, by using *Mwrank*, we can obtain the ranks of corresponding curves.

5. Results about the new family of curves

Remark 5.1. The elliptic curve in the form $y^2 = x(x-a^2)(x-b^2)$ for any pythagorean triples (a, b, c) is smooth, in fact $a \neq b$ and both are nonzero.

Remark 5.2. In the equation (4.1), let j be a constant and write (4.1), in the form (1.5). So a and b, are polynomials of i, and their degree are equal to 8 and 12. By [2], we have $r \leq 2\max\{3dega, 2degb\} = 48$

Lemma 5.3. *The elliptic curve in the form* (4.1) *has four points of order 2.*

Proof. It is clear that the points $P_1 = (0,0), P_2 = (a^2,0), P_3 = (b^2,0)$ are of order 2. Then $2E(Q) \simeq \frac{Z}{2Z} \oplus \frac{Z}{2Z}$. □

Theorem 5.4. *Let E be an elliptic curve defined over a field F, by the equation $y^2 = (x-\alpha)(x-\beta)(x-\gamma) = x^3 + ax^2 + bx + c$, where $Char(F) \neq 2$. For $(x', y') \in E(F)$, there exists $(x, y) \in E(F)$ with $2(x, y) = (x', y')$, if and only if $x' - \alpha$, $x' - \beta$, and $x' - \gamma$ are squares.*

Proof. ([4]. Th 4.1. pp.37). □

Theorem 5.5. *The elliptic curve in the form* (4.1) *doesn't have any point of order* 4.

Proof. Let $P = (x, y) \in E(Q)$, such that $4P = O$. Then one of following cases must be true.

$$2P = (0,0) \qquad or \quad 2P = (a^2,0) \quad or \qquad 2P = (b^2,0).$$

If $2P = (0,0)$, then $-a^2$ and $-b^2$, are squares, which is a contradiction. If $2P = (a^2,0)$, then $a^2 - b^2$ is a square. So we have, $a^2 - b^2 = d^2$ for some $d \in Z$ and $a^2 + b^2 = c^2$. Therefore $(\frac{a}{b})^2 - 1 = (\frac{d}{b})^2$ and $(\frac{a}{b})^2 + 1 = (\frac{c}{b})^2$. It turn out that 1 is a congruent number again a contradiction. The case $2P = (b^2,0)$ is similar. □

Corollary 5.6. *There is a no point of order* 8 *on* (4.1) .

Kubert [5], showed that if $y^2 = x(x+r)(x+s)$, with $r, s \neq 0$ and $s \neq r$, then the torsion subgroup is $\frac{Z}{2Z} \times \frac{Z}{2Z}$. So our family have $\frac{Z}{2Z} \times \frac{Z}{2Z}$ as torsion subgroup.

Lemma 5.7. *For each pythagorean triple (a, b, c), the elliptic Curve $y^2 = x(x - a^2)(x - b^2)$ has a positive rank.*

Proof. Choose $x = c^2$, then $P = (c^2, \pm abc)$. We show that for each (a, b, c), abc does not divide the *discriminant* D, where $D = a^4b^4(c^4 - 4a^2b^2)$. If $abc \mid a^4b^4(c^4 - 4a^2b^2)$ then $c \mid a^3b^3(c^4 - 4a^2b^2)$. Let p is a prime number such that $p \mid c$, then $p \mid -4a^2b^2$, but c is odd, then $p \neq 2$ so $p \mid a^2b^2$ and hence $p|a$ or $p|b$, which is a contradiction. So $p = (c^2, \pm abc)$ has integer coordinate in which $y = \pm abc$ does not divide D. Therefore by $Nagell - Lutz$ theorem P does not have finite order. This implies that $r \geq 1$. □

6. Numerical Results

After searching through 202461 curves, we found 12 curves with *selmer* 6. But unfortunately none of them had *rank* 6. Also we found 831 curves with *selmer* 5, leading to 52 curves of rank 5.
The first curve that generated by first pythagorean triple $(3, 4, 5)$ has *rank* 1.

In the following table, we listed the curves that have selmer equals to 6, without being able to compute their exact ranks with MWrank.

i	j	(a, b, c)	curve	bound
598	53	$(354795, 63388, 360413)$	$y^2 = x^3 - 129897530569x^2$ $+50578865085559061160 0x$	$4 \leq r \leq 6$
629	202	$(354837, 254116, 436445)$	$y^2 = x^3 - 190484238025x^2$ $+8130585454709316664464x$	$4 \leq r \leq 6$
760	113	$(564831, 171760, 590369)$	$y^2 = x^3 - 348535556161x^2$ $+9411982512955600953600x$	$4 \leq r \leq 6$
777	232	$(549905, 360528, 657553)$	$y^2 = x^3 - 432375947809x^2$ $+39305500949380532025600x$	$4 \leq r \leq 6$
801	560	$(328001, 897120, 955201)$	$y^2 = x^3 - 912408950401x^2$ $+86586744854271550694400x$	$1 \leq r \leq 6$
821	242	$(615477, 397364, 732605)$	$y^2 = x^3 - 536710086025x^2$ $+59813703564011517306384x$	$2 \leq r \leq 6$
861	788	$(120377, 1356936, 1362265)$	$y^2 = x^3 - 1855765930225x^2$ $+26681224725077190456384x$	$2 \leq r \leq 6$
890	457	$(583251, 813460, 1000949)$	$y^2 = x^3 - 1001898900601x^2$ $+225104091544539413571600x$	$2 \leq r \leq 6$
917	846	$(125173, 1551564, 1556605)$	$y^2 = x^3 - 2423019126025x^2$ $+37719046943947124807184x$	$4 \leq r \leq 6$
957	788	$(294905, 1508232, 1536793)$	$y^2 = x^3 - 2361732724849x^2$ $+197833836741502151361600x$	$2 \leq r \leq 6$
958	691	$(440283, 1323956, 1395245)$	$y^2 = x^3 - 1946708610025x^2$ $+339790269763746950924304x$	$1 \leq r \leq 6$
964	173	$(899367, 333544, 959225)$	$y^2 = x^3 - 920112600625x^2$ $+89987080452485248355904x$	$2 \leq r \leq 6$

TABLE 2. The curves with selmer-rank 6.

In the following table, we listed some curves which have rank 5.

n	i	j	(a, b, c)	curve	*rank*
1	65	58	$(861, 7540, 7589)$	$y^2 = x^3 - 57592921x^2$ $+42145284963600x$	5
2	206	73	$(37107, 30076, 47765)$	$y^2 = x^3 - 2281495225x^2$ $+1245523255531937424x$	5
3	219	122	$(33077, 53436, 62845)$	$y^2 = x^3 - 3949494025x^2$ $+3124065342026615184x$	5
4	221	74	$(43365, 32708, 54317)$	$y^2 = x^3 - 2950336489x^2$ $+2011808689365056400x$	5
5	226	197	$(12267, 89044, 89885)$	$y^2 = x^3 - 8079313225x^2$ $+1193125293288351504x$	5
6	277	148	$(54825, 81992, 98633)$	$y^2 = x^3 - 9728468689x^2$ $+20206925530689960000x$	5
7	291	130	$(67781, 75660, 101581)$	$y^2 = x^3 - 10318699561x^2$ $+26299568174145411600x$	5
8	298	241	$(30723, 143636, 146885)$	$y^2 = x^3 - 21575203225x^2$ $+19473940840993453584x$	5
9	305	146	$(71709, 89060, 114341))$	$y^2 = x^3 - 13073864281x^2$ $+40786150175724531600x$	5
10	325	132	$(88201, 85800, 123049)$	$y^2 = x^3 - 15141056401x^2$ $+57269262954257640000x$	5

TABLE 3. Some curves with ranks 5.

n	Independent points
1	$(\frac{57564577194761}{1008016}, \frac{29006793653594700125}{1012048064}),(\frac{165532287616200}{2745649}, \frac{505394258095121556600}{4549540393})$ $(\frac{6192906993}{64}, \frac{311795186829399}{512}),(\frac{24834332880}{121}, \frac{3321719539155360}{1331})$ $(341015696, 5742307020800)$
2	$(\frac{166618634504}{121}, \frac{311255416873240}{1331}),(\frac{12790926337}{9}, \frac{-153963331881884}{27})$ $(1862526649, 29434944424380),(\frac{14584697373888197298}{2226990481}, \frac{45953060323429949195929519458}{105093907788871})$ $(11173929032, 1060281679441544)$
3	$(\frac{1420783000225}{2704}, \frac{-3709951931018864055}{140608}),(\frac{34263881899795 46}{3150625}, \frac{-19862798666292714153406}{5592359375})$ $(\frac{3209176809789192}{1100401}, \frac{20777492819646247103496}{1154320649}),(\frac{5079795156916250}{1371241}, \frac{14550483032160729130 8950}{1605723211})$ $(11153906082, 964957876872066)$
4	$(1883980800, 2302931030400),(2049417864, 18414019508040)$ $(\frac{2442134720068225}{602176}, \frac{-758334011811429462386 25}{467288576}),(8778656250, -683241762498750)$ $(\frac{389025929026}{9}, \frac{-234351164774907530}{27})$
5	$(\frac{40247709912197}{724201}, \frac{-397145027493508897009 4}{616295051}),(\frac{14644921094163784}{1292769}, \frac{964386979747182474225400}{1469878353})$ $(\frac{87950467020096}{6889}, \frac{504745975500657035040}{571787}),(18277955208, 1851757920077688)$ $(42787752953, 7974645953968408)$
6	$(\frac{52434265914}{249001}, \frac{-2562930282129146180 10}{124251499}),(120296250, -47872494168750)$ $(6723284800, 3861958531200),(\frac{112595270161250}{16129}, \frac{17340008611175648875 0}{2048383})$ $(\frac{14340640706653}{361}, \frac{4758909704295045305 4}{6859})$
7	$(\frac{2676650962237850}{1394761}, \frac{-23023471487528264011025 0}{1647212741}),(\frac{22163879894522425}{5216656}, \frac{-5546287656665725432859 25}{11914842304})$ $(\frac{34346962133043282}{5997601}, \frac{57316484301139284256098}{14688124849}),(6253062480, 74048765888160)$ $(\frac{109261411840568520}{717409}, \frac{34892314618842917159456520}{607645423})$
8	$(\frac{730404089870769}{891136}, \frac{-37789359740568919672425}{841232384}),(\frac{5478549187165109}{6056521}, \frac{-39487422947402698353371 0}{14905098181})$ $(20665851602, 118667705326126),(\frac{73166967363875922}{2745649}, \frac{92362927560191302016290 86}{4549540393})$ $(51598853768, 8996724544134712)$
9	$(1837492490, -192369433165070),(2274211682, -192094032181618)$ $(\frac{3557867077800}{361}, \frac{20505067695974358 00}{6859})$ $(\frac{699532475085000}{32761}, \frac{12780541414500071841000}{5929741}), (\frac{831997800678440}{29929}, \frac{18315695665342299799960}{5177717})$
10	$(7819306560, 11947900423680),(\frac{947937694496}{121}, \frac{18954422023540640}{1331})$ $(7908659200, 23645902425600),(\frac{49352010853464722}{4977361}, \frac{2582386656676462513905118}{11104492391})$ $(\frac{6348468129250}{49}, \frac{-1506101738256255 0750}{343})$

TABLE 4. Independent points of curves of table 3.

i	j	(a,b,c)	curve	*rank*
26	17	$(387, 884, 965)$	$y^2 = x^3 - 931225x^2$ $+117037883664x$	4
43	24	$(1273, 2064, 2425)$	$y^2 = x^3 - 5880625x^2$ $+6903609110784x$	4
55	34	$(1869, 3740, 4181)$	$y^2 = x^3 - 17480761x^2$ $+48860938803600x$	4
63	40	$(2369, 5040, 5569)$	$y^2 = x^3 - 31013761x^2$ $+142557868857600x$	4
66	47	$(2147, 6204, 6565)$	$y^2 = x^3 - 43099225x^2$ $+177422080320144x$	4
71	58	$(1677, 8236, 8405)$	$y^2 = x^3 - 70644025x^2$ $+190765045779984x$	4
74	5	$(5451, 740, 5501)$	$y^2 = x^3 - 30261001x^2$ $+16271058387600x$	4
74	23	$(4947, 3404, 6005)$	$y^2 = x^3 - 36060025x^2$ $+283571724009744$	4
74	53	$(2667, 7844, 8285)$	$y^2 = x^3 - 68641225x^2$ $+437644224322704x$	4
78	35	$(4859, 5460, 7309)$	$y^2 = x^3 - 53421481x^2$ $+703848328419600x$	4

TABLE 5. Some curves with ranks 4.

i	j	(a, b, c)	curve	*rank*
13	6	$(133, 156, 205)$	$y^2 = x^3 - 42025x^2 + 430479504x$	3
13	10	$(69, 260, 269)$	$y^2 = x^3 - 72361x^2 + 321843600x$	3
19	6	$(325, 228, 397)$	$y^2 = x^3 - 157609x^2 + 5490810000x$	3
20	3	$(391, 120, 409)$	$y^2 = x^3 - 167281x^2 + 2201486400x$	3
21	8	$(377, 336, 505)$	$y^2 = x^3 - 255025x^2 + 16045795584x$	3
21	10	$(341, 420, 541)$	$y^2 = x^3 - 292681x^2 + 20511968400x$	3
4	3	$(7, 24, 25)$	$y^2 = x^3 - 625x^2 + 28224x$	2
5	2	$(21, 20, 29)$	$y^2 = x^3 - 841x^2 + 176400x$	2
7	4	$(33, 56, 65)$	$y^2 = x^3 - 4225x^2 + 3415104x$	2
8	1	$(63, 16, 65)$	$y^2 = x^3 - 4225x^2 + 1016064x$	2
9	2	$(77, 36, 85)$	$y^2 = x^3 - 7225x^2 + 7683984x$	2
2	1	$(3, 4, 5)$	$y^2 - 25x^2 + 144x$	1
3	2	$(5, 12, 13)$	$y^2 = x^3 - 169x^2 + 3600x$	1
4	1	$(15, 8, 17)$	$y^2 = x^2 - 289x^2 + 14400x$	1
5	4	$(9, 40, 41)$	$y^2 = x^3 - 1681x^2 + 129600x$	1
6	1	$(35, 12, 37)$	$y^2 = x^3 - 1369x^2 + 176400x$	1

TABLE 6. Some curves with rank 3,2, and 1.

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$$a_1Y = 3X^2 + 2a_2X + a_4 \quad and \quad 2Y + a_1X + a_3 = 0$$

cannot be simultaneously satisfied by any $(x, y) \in E(\overline{F})$.

If $Char(F) \neq 2$, we can reduce (1.1) to the following form

$$Y^2 = X^3 + aX^2 + bX + C \tag{1.3}$$

with the *discriminant* :

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If furthermore, the $Char(F)$ does not divide 6, then we get the simplest form of

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Remark 1.1. The elliptic curve is smooth if and only if $D \neq 0$ [8].

2. Elliptic curves over Q

Mordell proved that on a rational elliptic curve, the rational points form a finitely generated abelian group, which is denoted by $E(Q)$ [4]. Here we can apply the structure theorem for the finitely generated abelian groups to $E(Q)$ to obtain a decomposition of $E(Q) \cong Z^r \times Tors_E(Q)$, where r is an integer called the *rank* of E and $Tors_E(Q)$ is the finite abelian group consisting of all the elements of finite order in $E(Q)$.
In 1976, *Barry Mazur*, proved the following fundamental result. The torsion group of every elliptic curve is one of the following 15 cases :

$$\frac{Z}{mZ} \qquad m = 1, 2, 3, ..., 10, 12$$
$$\frac{Z}{2Z} \oplus \frac{Z}{mZ} \quad m = 2, 4, 6, 8. \tag{2.1}$$

This shows that there is no points of order 11, and any $n \geq 13$.
There is an important theorem proved by *Nagell* and *Lutz*, which tells us how to find all the rational points of finite order.

Theorem 2.1. *(Nagell-Lutz) Let E be given by $y^2 = x^3 + ax^2 + bx + c$ with $a, b, c \in Z$. Let $P = (x, y) \in E(Q)$. Suppose P has finite order, Then $x, y \in Z$ and either $y = 0$ or $y^2|D$.*

Proof. ([7] . *pp* . 56). □

Theorem 2.2. *Let E be given by $y^2 = x^3 + ax^2 + bx + c$ and, $P = (x, y) \in E(Q)$. P has an order 2 if and only if $y = 0$.*

Proof. ([8]. *pp* .77) . □

On the other hand, it is not known which values of *rank* r are possible. The current record is an example of elliptic curve over Q with *rank* ≥ 28 found by *Elkies* in may 2006 [2].

In this Paper we first introduce a family of elliptic curves over Q and show that they have positive rank, then search for the largest ranks possible.

3. Pythagorean triples

A primitive pythagorean triple is a triple of numbers (a, b, c) so that a , b and c have no common divisors and satisfy

$$a^2 + b^2 = c^2. \tag{3.1}$$

It's not hard to prove that if one of a or b is odd then the other is even, then c is always odd.
In general , we can generate (a, b, c) by the following relations:

$$a = i^2 - j^2 \qquad b = 2ij \qquad c = i^2 + j^2 \tag{3.2}$$

where *gcd* $(i, j) = 1$ and i, j have oppositive parity.

The following table gives all possible triples with $i, j < 10$.

i	j	$a = i^2 - j^2$	$b = 2ij$	$c = i^2 + j^2$	(a, b, c)
2	1	3	4	5	$(3, 4, 5)$
3	2	5	12	13	$(5, 12, 13)$
4	1	15	8	17	$(15, 8, 17)$
4	3	7	24	25	$(7, 24, 25)$
5	2	21	20	29	$(21, 20, 29)$
5	4	9	40	41	$(9, 40, 41)$
6	1	35	12	37	$(35, 12, 37)$
6	5	11	60	61	$(11, 60, 61)$
7	2	45	28	53	$(45, 28, 53)$
7	4	33	56	65	$(33, 56, 65)$
7	6	13	84	85	$(13, 84, 85)$
8	1	63	16	65	$(63, 16, 65)$
8	3	55	48	73	$(55, 48, 73)$
8	5	39	80	89	$(39, 80, 89)$
8	7	15	80	113	$(15, 80, 113)$
9	2	77	36	85	$(77, 36, 85)$
9	4	65	72	97	$(65, 72, 97)$
9	8	17	144	145	$(17, 144, 145)$

TABLE 1. Generating the primitive pythagorean triples with $i, j < 10$

4. Structure Of The Curves

First we generate a list of primitive pythagorean triples (a, b, c) with $i, j \leq 1000$. This yields a list of 202461 triples. Each (a, b, c) gives rise to the elliptic curve in the form

$$y^2 = x(x - a^2)(x - b^2). \tag{4.1}$$

Then we compute the $2-$ *selmer ranks* of these curves as upper bounds on the *Mordell* $-$ *Weil ranks*, finally, by using *Mwrank*, we can obtain the ranks of corresponding curves.

5. Relation Between Euler's Concordant forms and elliptic curves

In 1780, Euler asked for a classification of those pairs of distinct non-zero integers M and N for which there are integers solutions (x, y, t, z) with $xy \neq 0$ to the system of equation

$$\begin{aligned} x^2 + My^2 = t^2 \\ x^2 + Ny^2 = z^2. \end{aligned} \tag{5.1}$$

One can consider Euler's problem as the problem of the study of the elliptic curve over Q. i.e :

$$E_Q(M, N): \quad y^2 = x^3 + (M + N)x^2 + MNx. \tag{5.2}$$

A solution to (5.1) is primitive, if x, y, t, and z are positive integers and $gcd(x, y) = 1$. If $E_Q(M, N)$ has positive rank, then there are infinity many primitive integer solutions to (5.1) [5]. If $E_Q(M, N)$ has rank 0, then (5.1) has a solution if and only if the torsion group is

$$\frac{Z}{2Z} \oplus \frac{Z}{8Z} \quad or \quad \frac{Z}{2Z} \oplus \frac{Z}{6Z}.$$

We can let the $gcd(M, N)$ be a square-free integer, also we can show that $E_Q(M, N) \simeq E_Q(-M, N - M) \simeq E_Q(-N, M - N)$. Therefore without loss of generality assume that M and N are both positive integers.
So in (5.2), if we let $M = -a^2$ and $N = -b^2$, where $a^2 + b^2 = c^2$, then

$$E_Q(-a^2, -b^2): \quad y^2 = x^3 + (-a^2 - b^2)x^2 + a^2b^2x = x(x - a^2)(x - b^2)$$

which is in the form of (4.1). Therefore if we can prove (4.1) has a positive rank or has either a torsion group of $\frac{Z}{2Z} \oplus \frac{Z}{8Z} \quad or \quad \frac{Z}{2Z} \oplus \frac{Z}{6Z}$, then it turns out that in this case (5.1), has infinity many solutions.
But as we shall see, (4.1) has the torsion group of $\frac{Z}{2Z} \oplus \frac{Z}{2Z}$. To prove that our family of elliptic curves has $\frac{Z}{2Z} \oplus \frac{Z}{2Z}$ as a torsion group, among other things, we need to use the following theorem too.

Theorem 5.1. *The torsion subgroup of $E_Q(M, N)$ are uniquely determined by the following four cases:*
i) The torsion subgroup of $E_Q(M, N)$ contains $\frac{Z}{2Z} \oplus \frac{Z}{4Z}$ if M and N are both squares, or $-M$ and $N - M$ are both squares, or if $-N$ and $M - N$ are both squares.

ii) The torsion subgroup of $E_Q(M, N)$ is $\frac{Z}{2Z} \oplus \frac{Z}{8Z}$ if there exists a non-zero integer d such that $M = d^2u^4$ and $N = d^2v^4$, or $M = -d^2v^4$ and $N = d^2(u^4 - v^4)$, or $M = d^2(u^4 - v^4)$ and $N = -d^2v^4$ where (u, v, w) forms a pythagorean triple.

iii) The torsion subgroup of $E_Q(M, N)$ is $\frac{Z}{2Z} \oplus \frac{Z}{6Z}$ if there exists integers a and b such that $\frac{a}{b} \notin \{-2, -1, \frac{-1}{2}, 0, 1\}$ and $M = a^4 + 2a^3b$ and $N = b^4 + 2ab^3$.

iv) In all other cases, the torsion subgroup of $E_Q(M, N)$ is $\frac{Z}{2Z} \oplus \frac{Z}{2Z}$

Proof. ([6]) □

6. Results About The New Family Of Curves

Remark 6.1. For any pythagorean triple (a, b, c),the elliptic curve in the form $y^2 = x(x - a^2)(x - b^2)$ is smooth. In fact $a \neq b$ and both are nonzero.

Remark 6.2. In [3], Fouvry and Pomykala lead to an interesting result which is following. Let E be an elliptic curve in the form

$$y^2 = x^3 + a(t)x + b(t) \tag{6.1}$$

where $a(t),\ b(t) \in Z[t]$. Then the average rank of E is bounded by $2\max\{3deg\, a, 2deg\, b\}$. Therefore if we change (4.1) to the (5.1) and let one of the i or j be constant, we would have a and b the polynomials with degree $8, 12$. So we have $r \leq 2\max\{3dega, 2degb\} = 48$.

Lemma 6.3. *The elliptic curve in the form* (4.1) *has four points of order 2.*

Proof. It is clear that the points $P_1 = (0,0), P_2 = (a^2,0), P_3 = (b^2,0)$ are of order 2. Then $2E(Q) \simeq \frac{Z}{2Z} \oplus \frac{Z}{2Z}$. □

Theorem 6.4. *Let E be an elliptic curve defined over a field F, by the equation $y^2 = (x-\alpha)(x-\beta)(x-\gamma) = x^3 + ax^2 + bx + c$, where $Char(F) \neq 2$. For $(x', y') \in E(F)$, there exists $(x, y) \in E(F)$ with $2(x, y) = (x', y')$, if and only if $x' - \alpha$, $x' - \beta$, and $x' - \gamma$ are squares.*

Proof. ([4]. Th 4.1. pp.37). □

Theorem 6.5. *The elliptic curve in the form* (4.1) *doesn't have any point of order* 4.

Proof. Let $P = (x, y) \in E(Q)$, such that $4P = O$. Then one of following cases must be true.

$$2P = (0,0) \quad \text{or} \quad 2P = (a^2,0) \quad \text{or} \quad 2P = (b^2,0).$$

If $2P = (0,0)$, then $-a^2$ and $-b^2$, are squares, which is a contradiction. If $2P = (a^2, 0)$, then $a^2 - b^2$ is a square. So we have, $a^2 - b^2 = d^2$ for some $d \in Z$ and $a^2 + b^2 = c^2$. Therefore $(\frac{a}{b})^2 - 1 = (\frac{d}{b})^2$ and $(\frac{a}{b})^2 + 1 = (\frac{c}{b})^2$. It turns out that 1 is a congruent number again a contradiction. The case $2P = (b^2, 0)$ is similar. □

Corollary 6.6. *There is a no point of order* 8 *on* (4.1) .

Theorem 6.7. *The elliptic curve in the form* (4.1) *does not have any point of order* 6.

Proof. We prove this by theorem (5.1). Let $M = -a^2$ and $N = -b^2$ and without loss of generality assume that $a^2 < b^2$. Because $E_Q(M,N) \simeq E_Q(-N,\ M-N)$, we continue the proof with $E_Q(-N,\ M-N)$ which in this case both of the $-N$ and $M-N$ are positive integers. Let there exist integers A and B such that $\frac{A}{B} \notin \{-2,$-$1,\ \frac{-1}{2},\ 0,\ 1\}$ and $-N = b^2 = A^4 + 2A^3B$ and $M - N = b^2 - a^2 = B^4 + 2AB^3$. Let b is a even number, so A is as well and since $b^2 - a^2$ is odd, then B must be odd. Since $gcd\ (a,b) = 1$ we have $gcd\ (A,B) = 1$. $b^2 = A^3(A + 2B)$ so $a = t^2$ and $A + 2B = s^2$ where $t, s \in Z$. Because A is even, so $A + 2B = s^2$ is as well, thus $A + 2B \equiv 0(mod\ 4)$, in other hand A is even and square, thus $A \equiv\ 0\ (mod\ 4)$, which means that $2 \mid B$ which is a contradiction.
Now let b is odd, we conclude that both of A and B are odd. So $A + 2B = s^2$ is odd and then $s^2 \equiv 1(mod\ 4)$ and $a \equiv t^2 \equiv 1(mod\ 4)$, then we have $B \equiv 0(mod\ 2)$, which is again a contradiction. □

Lemma 6.8. *For each pythagorean triple (a, b, c), the elliptic Curve $y^2 = x(x - a^2)(x - b^2)$ has a positive rank.*

Proof. Choose $x = c^2$, then $P = (c^2, \pm abc)$. We show that for each (a, b, c), abc does not divide the $discriminant$ D, where $D = a^4b^4(c^4 - 4a^2b^2)$. If $abc \mid a^4b^4(c^4 - 4a^2b^2)$ then $c \mid a^3b^3(c^4 - 4a^2b^2)$. Let p is a prime number such that $p \mid c$, then $p \mid -4a^2b^2$, but c is odd, then $p \neq 2$ so $p \mid a^2b^2$ and hence $p|a$ or $p|b$, which is a contradiction. So $p = (c^2, \pm abc)$ has integer coordinate in which $y = \pm abc$ does not divide D. Therefore by $Nagell - Lutz$ theorem P does not have finite order. This implies that $r \geq 1$. □

Corollary 6.9. *In the case $M = -a^2$ and $N = -b^2$, where (a, b, c) is a Pythagorean triple, the Euler's concordant forms has a infinitely many primitive solution.*

7. Numerical Results

After searching through 202461 curves, we found 12 curves with *selmer* 6. But none of them had *rank* 6. Also we found 834 curves with *selmer* 5, leading to 53 curves of rank 5.
The first curve that generated by first pythagorean triple $(3, 4, 5)$ has *rank* 1.

In the following table, we have summarized the results of our computation.

Rank	number	percent
$rank = 1$	45847	22.6
$rank = 2$	16690	8.2
$rank = 3$	6699	3.3
$rank = 4$	948	0.4
$rank = 5$	53	0.02
$1 \leq rank \leq 2$	73204	36.1
$1 \leq rank \leq 3$	41381	20.4
$1 \leq rank \leq 4$	5906	2.9
$1 \leq rank \leq 5$	384	0.1
$1 \leq rank \leq 6$	2	0.0009
$2 \leq rank \leq 3$	6250	3
$2 \leq rank \leq 4$	4507	2.2
$2 \leq rank \leq 5$	100	0.04
$2 \leq rank \leq 6$	5	0.002
$3 \leq rank \leq 4$	183	0.09
$3 \leq rank \leq 5$	296	0.14
$3 \leq rank \leq 6$	0	0
$4 \leq rank \leq 5$	1	0.0004
$4 \leq rank \leq 6$	5	0.002
$5 \leq rank \leq 6$	0	0

Table 2. The results of computation.

In the table 3, we have listed the curves that have selmer equals to 6, without being able to compute their exact ranks with MWrank.

i	j	(a, b, c)	curve	bound
598	53	$(354795, 63388, 360413)$	$y^2 = x^3 - 129897530569x^2$ $+5057886508555906116 00x$	$4 \le r \le 6$
629	202	$(354837, 254116, 436445)$	$y^2 = x^3 - 190484238025x^2$ $+8130585454709316664464x$	$4 \le r \le 6$
760	113	$(564831, 171760, 590369)$	$y^2 = x^3 - 348535556161x^2$ $+9411982512955600953600x$	$4 \le r \le 6$
777	232	$(549905, 360528, 657553)$	$y^2 = x^3 - 432375947809x^2$ $+39305500949380532025600x$	$4 \le r \le 6$
801	560	$(328001, 897120, 955201)$	$y^2 = x^3 - 912408950401x^2$ $+86586744854271550694400x$	$1 \le r \le 6$
821	242	$(615477, 397364, 732605)$	$y^2 = x^3 - 536710086025x^2$ $+59813703564011517306384x$	$2 \le r \le 6$
861	788	$(120377, 1356936, 1362265)$	$y^2 = x^3 - 1855765930225x^2$ $+26681224725077190456384x$	$2 \le r \le 6$
890	457	$(583251, 813460, 1000949)$	$y^2 = x^3 - 1001898900601x^2$ $+225104091544539413571600x$	$2 \le r \le 6$
917	846	$(125173, 1551564, 1556605)$	$y^2 = x^3 - 2423019126025x^2$ $+37719046943947124807184x$	$4 \le r \le 6$
957	788	$(294905, 1508232, 1536793)$	$y^2 = x^3 - 2361732724849x^2$ $+197833836741502151361600x$	$2 \le r \le 6$
958	691	$(440283, 1323956, 1395245)$	$y^2 = x^3 - 1946708610025x^2$ $+339790269763746950924304x$	$1 \le r \le 6$
964	173	$(899367, 333544, 959225)$	$y^2 = x^3 - 920112600625x^2$ $+89987080452485248355904x$	$2 \le r \le 6$

TABLE 3. The curves with selmer-rank 6.

Table4, shows some curves which rank 5.

n	i	j	(a, b, c)	curve	*rank*
1	65	58	$(861, 7540, 7589)$	$y^2 = x^3 - 57592921x^2$ $+42145284963600x$	5
2	206	73	$(37107, 30076, 47765)$	$y^2 = x^3 - 2281495225x^2$ $+1245523255531937424x$	5
3	219	122	$(33077, 53436, 62845)$	$y^2 = x^3 - 3949494025x^2$ $+3124065342026615184x$	5
4	221	74	$(43365, 32708, 54317)$	$y^2 = x^3 - 2950336489x^2$ $+2011808689365056400x$	5
5	226	197	$(12267, 89044, 89885)$	$y^2 = x^3 - 8079313225x^2$ $+1193125293288351504x$	5
6	277	148	$(54825, 81992, 98633)$	$y^2 = x^3 - 9728468689x^2$ $+20206925530689960000x$	5
7	291	130	$(67781, 75660, 101581)$	$y^2 = x^3 - 10318699561x^2$ $+26299568174145411600x$	5
8	298	241	$(30723, 143636, 146885)$	$y^2 = x^3 - 21575203225x^2$ $+19473940840993453584x$	5
9	305	146	$(71709, 89060, 114341))$	$y^2 = x^3 - 13073864281x^2$ $+40786150175724531600x$	5
10	325	132	$(88201, 85800, 123049)$	$y^2 = x^3 - 15141056401x^2$ $+57269262954257640000x$	5

TABLE 4. Some curves with rank 5.

In the following table, we have listed the independent points of the curves of table 4

n	Independent points
1	$(\frac{57564577194761}{1008016}, \frac{29006793653594700125}{1012048064}),(\frac{165532287616200}{2745649}, \frac{5053942580951215566000}{4549540393})$ $(\frac{6192906993}{64}, \frac{311795186829399}{512}),(\frac{24834332880}{121}, \frac{3321719539155360}{1331})$ $(341015696, 5742307020800)$
2	$(\frac{166618634504}{121}, \frac{311255416873240}{1331}),(\frac{12790926337}{9}, \frac{-153963331881884}{27})$ $(1862526649, 29434944424380),(\frac{14584697373888197298}{2226990481}, \frac{45953060323429949195929519458}{105093907788871})$ $(11173929032, 1060281679441544)$
3	$(\frac{1420783000225}{2704}, \frac{-3709951931018864055}{140608}),(\frac{3426388189979546}{3150625}, \frac{-19862798666292714153406}{5592359375})$ $(\frac{3209176809789192}{1100401}, \frac{20777492819646247103496}{1154320649}),(\frac{5079795156916250}{1371241}, \frac{145504830321607291308950}{1605723211})$ $(11153906082, 964957876872066)$
4	$(1883980800, 2302931030400),(2049417864, 18414019508040)$ $(\frac{2442134720068225}{602176}, \frac{-75833401181142946238625}{467288576}),(8778656250, -683241762498750)$ $(\frac{389025929026}{9}, \frac{-234351164774907530}{27})$
5	$(\frac{40247709912197}{724201}, \frac{-3971450274935088970094}{616295051}),(\frac{14644921094163784}{1292769}, \frac{964386979747182474225400}{1469878353})$ $(\frac{87950467020096}{6889}, \frac{504745975500657035040}{571787}),(18277955208, 1851757920077688)$ $(42787752953, 7974645953968408)$
6	$(\frac{52434265914}{249001}, \frac{-256293028212914618010}{124251499}),(120296250, -47872494168750)$ $(6723284800, 3861958531200),(\frac{112595270161250}{16129}, \frac{173400086111756488750}{2048383})$ $(\frac{14340640706653}{361}, \frac{47589097042950453054}{6859})$
7	$(\frac{2676650962237850}{1394761}, \frac{-230234714875282640110250}{1647212741}),(\frac{22163879894522425}{5216656}, \frac{-554628765666572543285925}{11914842304})$ $(\frac{34346962133043282}{5997601}, \frac{57316484301139284256098}{14688124849}),(6253062480, 74048765888160)$ $(\frac{109261411840568520}{717409}, \frac{34892314618842917159456520}{607645423})$
8	$(\frac{730404089870769}{891136}, \frac{-37789359740568919672425}{841232384}),(\frac{5478549187165109}{6056521}, \frac{-394874229474026983533710}{14905098181})$ $(20665851602, 118667705326126),(\frac{73166967363875922}{2745649}, \frac{9236292756019130201629086}{4549540393})$ $(51598853768, 8996724544134712)$
9	$(1837492490, -192369433165070),(2274211682, -192094032181618)$ $(\frac{3557867077800}{361}, \frac{2050506769597435800}{6859})$ $(\frac{699532475085000}{32761}, \frac{12780541414500071841000}{5929741}), (\frac{831997800678440}{29929}, \frac{18315695665342299799960}{5177717})$
10	$(7819306560, 11947900423680),(\frac{947937694496}{121}, \frac{18954422023540640}{1331})$ $(7908659200, 23645902425600),(\frac{49352010853464722}{4977361}, \frac{2582386656676462513905118}{11104492391})$ $(\frac{6348468129250}{49}, \frac{-15061017382562550750}{343})$

TABLE 5. Independent points of curves of table 3.

i	j	(a,b,c)	curve	*rank*
26	17	$(387, 884, 965)$	$y^2 = x^3 - 931225x^2$ $+117037883664x$	4
43	24	$(1273, 2064, 2425)$	$y^2 = x^3 - 5880625x^2$ $+6903609110784x$	4
55	34	$(1869, 3740, 4181)$	$y^2 = x^3 - 17480761x^2$ $+48860938803600x$	4
63	40	$(2369, 5040, 5569)$	$y^2 = x^3 - 31013761x^2$ $+142557868857600x$	4
66	47	$(2147, 6204, 6565)$	$y^2 = x^3 - 43099225x^2$ $+177422080320144x$	4
71	58	$(1677, 8236, 8405)$	$y^2 = x^3 - 70644025x^2$ $+190765045779984x$	4
74	5	$(5451, 740, 5501)$	$y^2 = x^3 - 30261001x^2$ $+16271058387600x$	4
74	23	$(4947, 3404, 6005)$	$y^2 = x^3 - 36060025x^2$ $+283571724009744$	4
74	53	$(2667, 7844, 8285)$	$y^2 = x^3 - 68641225x^2$ $+437644224322704x$	4
78	35	$(4859, 5460, 7309)$	$y^2 = x^3 - 53421481x^2$ $+703848328419600x$	4

TABLE 6. Some curves with rank 4.

i	j	(a,b,c)	curve	*rank*
13	6	$(133,156,205)$	$y^2 = x^3 - 42025x^2 + 430479504x$	3
13	10	$(69,260,269)$	$y^2 = x^3 - 72361x^2 + 321843600x$	3
19	6	$(325,228,397)$	$y^2 = x^3 - 157609x^2 + 5490810000x$	3
20	3	$(391,120,409)$	$y^2 = x^3 - 167281x^2 + 2201486400x$	3
21	8	$(377,336,505)$	$y^2 = x^3 - 255025x^2 + 16045795584x$	3
21	10	$(341,420,541)$	$y^2 = x^3 - 292681x^2 + 20511968400x$	3
4	3	$(7,24,25)$	$y^2 = x^3 - 625x^2 + 28224x$	2
5	2	$(21,20,29)$	$y^2 = x^3 - 841x^2 + 176400x$	2
7	4	$(33,56,65)$	$y^2 = x^3 - 4225x^2 + 3415104x$	2
8	1	$(63,16,65)$	$y^2 = x^3 - 4225x^2 + 1016064x$	2
9	2	$(77,36,85)$	$y^2 = x^3 - 7225x^2 + 7683984x$	2
2	1	$(3,4,5)$	$y^2 - 25x^2 + 144x$	1
3	2	$(5,12,13)$	$y^2 = x^3 - 169x^2 + 3600x$	1
4	1	$(15,8,17)$	$y^2 = x^2 - 289x^2 + 14400x$	1
5	4	$(9,40,41)$	$y^2 = x^3 - 1681x^2 + 129600x$	1
6	1	$(35,12,37)$	$y^2 = x^3 - 1369x^2 + 176400x$	1

TABLE 7. Some curves with rank 3,2, and 1.

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